Kochen-Specker theorem and games

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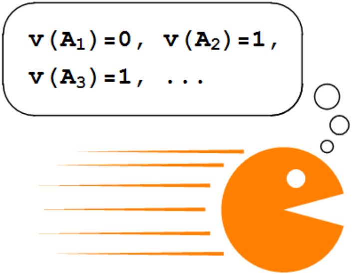
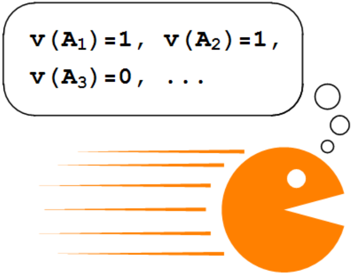
Hidden variables





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In this talk

We will consider proofs of several versions of Kochen-Specker theorem and games that are based on these proofs.

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Measurement described by an observable

Observable M is a Hermitian operator. If

$$M = \sum \lambda P_\lambda$$

is a spectral decomposition of M, then M defines a projective measurement in the following way:

- the outcome of the measurement is an eigenvalue λ of M,
- the state collapses to the corresponding eigenspace P_λ.

Commuting observables

Definition

Observables A and B are said to commute if

$$AB = BA$$

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Theorem

If mutually commuting observables $A_1, A_2, \ldots, A_n$ satisfy some functional identity

$$f(A_1, A_2, \ldots, A_n) = 0,$$

then the values assigned to them in an individual system must also be related by

$$f\Big(v(A_1), v(A_2), \ldots, v(A_n)\Big) = 0$$

Kochen-Specker theorem (3 dimensional version)

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Consequences of Kochen-Specker theorem

Every non-contextual hidden variables theory is inconsistent with quantum mechanics formalism.

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- Observable S_v measures the square of spin component of a spin 1 particle along direction $v \in \mathbb{R}^3$

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- The outcomes (eigenvalues) of the measurement S_v are 1 or 0
- If $\{u, v, w\}$ are mutually orthogonal vectors in $\mathbb{R}^3$, then
  1. $\{S_u, S_v, S_w\}$ is a set of mutually commuting observables
  2. $S_u + S_v + S_w = 2I$

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  1. $\{S_u, S_v, S_w\}$ is a set of mutually commuting observables
  2. $S_u + S_v + S_w = 2I \implies v(S_u) + v(S_v) + v(S_w) = 2.$

The task of proving Kochen-Specker theorem can be reduced to the following problem

Find a set of vectors in $\mathbb{R}^3$ for which it is impossible to assign "0" and "1" (outcomes of observables S_v) so that in each set of three mutually orthogonal vectors "1" is assigned to exactly two of them.

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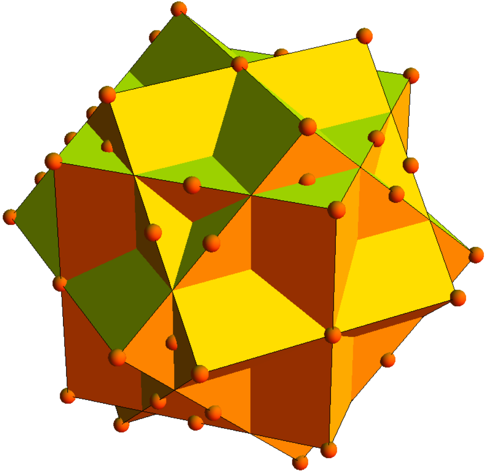
1. Kochen and Specker (1967) found the required set with 117 vectors
2. Later Conway and Kochen reduced the set to 31 vectors
3. Peres (1991) found the required set with 33 vectors (with nice symmetries)

Magic configuration



Although it is not obvious, this set satisfies the required property.

M.C.Escher "Waterfall"

Kochen-Specker game

Setting of the game

- Alice and Bob plays against verifier

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- No communication between Alice and Bob after the input from verifier is received

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As always we will see that entanglement turns out to be the key trick in quantum strategy.

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- Verifier chooses three mutually orthogonal vectors v_i, v_j, v_k from the set V. He asks
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  - Parity rule: "1" gets assigned to exactly two of the three vectors
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- Alice and Bob cannot always win if they use classical strategy as this would lead to violation of KS theorem.
- Yet they can win using quantum strategy with entanglement.

Quantum strategy for KS game

Alice and Bob share the state $|\Psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$.

1. Alice measures her qutrit with POVM $\{|v_i\rangle\langle v_i|, |v_j\rangle\langle v_j|, |v_k\rangle\langle v_k|\}$. She assigns "0" to the vector corresponding to the outcome of her measurement and "1" to the rest two vectors.
2. Bob measures with POVM $\{|v_l\rangle\langle v_l|, I - |v_l\rangle\langle v_l|\}$. He assigns "0" to vector v_l if the state collapses to $|v_l\rangle$ and "1" if otherwise.

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We need to check whether parity and consistency rules are always satisfied.

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- 4 dimensional Hilbert space corresponds to two qubit system
- Again we will construct a set of observables that satisfy some functional identities that cannot be satisfied by the values assigned to them.

Magic square

	X	Y	Z
X	I	iZ	−iY
Y	−iZ	I	iX
Z	iY	−iX	I

Multiplication of Pauli matrices.

−I	−I	−I	
$I\otimes Z$	$Z\otimes I$	$Z\otimes Z$	I
$X\otimes I$	$I\otimes X$	$X\otimes X$	I
$-X\otimes Z$	$-Z\otimes X$	$Y\otimes Y$	I

Magic square.

- Observables on each row and column are mutually commuting.

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$-I$	$-I$	$-I$	

Magic square.

- Observables on each row and column are mutually commuting.
- It is impossible to fill in the outcomes of observables so that functional identities are satisfied.

Game

- Verifier asks Alice to fill in some row and Bob to fill some column with "1" or "-1"

$-I$	$-I$	$-I$	
$I\otimes Z$	$Z\otimes I$	$Z\otimes Z$	I
$X\otimes I$	$I\otimes X$	$X\otimes X$	I
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Game

- Verifier asks Alice to fill in some row and Bob to fill some column with “1” or “-1”
- Alice and Bob win if
  - Parity rule The parity of “-1” is even for Alice and odd for Bob
  - Consistency rule Alice and Bob assign the same value to the intersection

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- There is no perfect classical strategy.

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Quantum strategy

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- Alice (Bob) measures her part of $|\Psi\rangle$ with the observables on the corresponding row (column) and gives verifier the outcomes of her measurement.

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We have to check whether parity and consistency rules hold.

Consistency rule verification

- Let $\mathcal{B} = \{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$ be a basis of Alice's state space (2 qubits) and $\overline{\mathcal{B}} = \{\overline{|b_1\rangle}, \overline{|b_2\rangle}, \overline{|b_3\rangle}, \overline{|b_4\rangle}\}$ be a basis of Bob's state space.

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- It turns out that $|\Psi\rangle = \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)^{\otimes 2}$ in these basis can be written as:

$$|\Psi\rangle = \frac{1}{4}\left(|b_1\rangle\overline{|b_1\rangle} + |b_2\rangle\overline{|b_2\rangle} + |b_3\rangle\overline{|b_3\rangle} + |b_4\rangle\overline{|b_4\rangle}\right)$$

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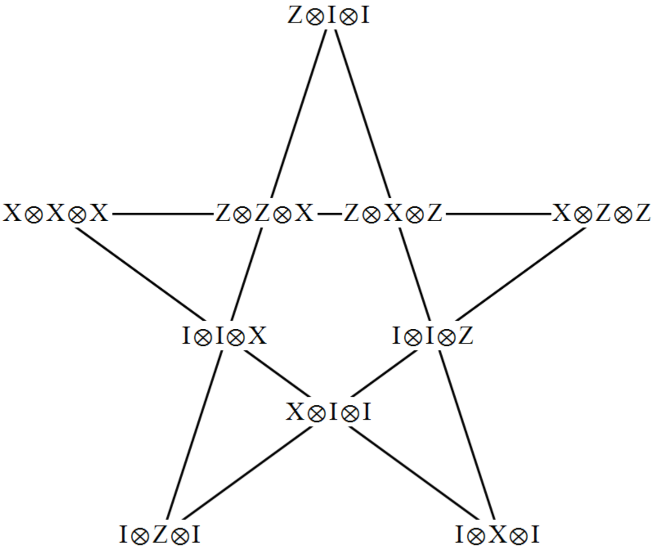
- Also it can be shown that the eigenvectors of observables being measured are real, therefore $\mathcal{B} = \overline{\mathcal{B}}$ and Bob will get the same outcome as Alice.

Magic star



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